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On the Existence of Generalized Inverses

GARY W. HANSEN AND DONALD W. ROBINSON
Brigham Young University
Provo, Utah

Recommended by Hans Schwerdtfeger

ABSTRACT

Conditions are given for the existence of a generalized inverse of a ring morphism, and the results are related to the theory of semi-simple Artinian rings. Necessary and sufficient conditions are also given for the existence of a generalized inverse of a morphism on real inner product spaces.

The concept of a generalized inverse has rated considerable attention during the past few years. In the setting of integral and differential operators, the idea dates from a 1903 paper of I. Fredholm [7], followed by a 1912 paper of W. A. Hurwitz [14]. The algebraic case was first considered by E. H. Moore [16, 17] in 1920. Since then the notion has been studied independently from various points of view by J. von Neumann [18], C. L. Siegel [25], K. O. Friedrichs [8], A. Bjerhammer [3], and R. Baer [2]. Recently numerous papers have appeared on the subject and several applications have been made. (See [4] for an extensive bibliography; also [5, 6, 10–13, 21–24, 26–28].)

In dealing with vector spaces of finite dimension, the existence of generalized inverses is always assured. In general, however, this need not be the case. In this note we make two observations. The first investigates the existence of generalized inverses in the setting of modules over a ring, and relates these results to the well-known theory of semi-simple Artinian rings. The second is concerned with generalized inverses in arbitrary inner product spaces. As has been previously noted in several special cases (see for example [4, 5, 20, 27]), generalized inverses may be used to obtain "best approximate" or "virtual" solutions to linear systems. We show that the existence of the "natural" generalized inverse in this case is in fact equivalent to the existence of "extremal" virtual solutions.

Let R be an associative ring with an identity and let E and F be (left) modules over R. (See for example [15, p. 1].)

DEFINITION. An R-morphism $\varphi\colon E \to F$ is said to be regular provided that there exists an R-morphism $\psi\colon F \to E$ such that

$$\varphi\psi\varphi = \varphi.$$

As a first fact we prove the following theorem.

THEOREM 1. *If $\varphi\colon E \to F$ is an R-morphism, then the following statements are equivalent*:

(1.1) *φ is regular.*

(1.2) *The kernel and image of φ are direct summands of E and F, respectively.*

Proof. Let φ be regular with $\varphi\psi\varphi = \varphi$. Since $\varphi\psi$ and $\psi\varphi$ are both idempotent,

$$E = \operatorname{Im}\varphi\psi \oplus \operatorname{Im}(\iota_E - \varphi\psi), \qquad F = \operatorname{Im}\psi\varphi \oplus \operatorname{Im}(\iota_F - \psi\varphi),$$

where ι_E and ι_F are the identity morphisms on E and F, respectively. Furthermore, since it is easily established that

$$\operatorname{Ker}\varphi = \operatorname{Im}(\iota_E - \varphi\psi), \qquad \operatorname{Im}\varphi = \operatorname{Im}\psi\varphi,$$

it follows that (1.1) implies (1.2).

Conversely, suppose $E = \operatorname{Ker}\varphi \oplus M$ and $F = \operatorname{Im}\varphi \oplus N$ for R-submodules M and N of E and F, respectively. Let $\pi_2\colon E \to M$ and $\pi_1\colon F \to \operatorname{Im}\varphi$ be the natural projections, let $\iota_2\colon M \to E$ and $\iota_1\colon \operatorname{Im}\varphi \to F$ be the natural injections, and let $\theta = \iota_2\varphi\pi_1$. Schematically, we have the following diagram:

$$\begin{array}{ccc}
\operatorname{Ker}\varphi \oplus M & \xrightarrow{\varphi} & \operatorname{Im}\varphi \oplus N, \\
\pi_2 \downarrow\uparrow \iota_2 & & \iota_1 \uparrow\downarrow \pi_1 \\
M & \xrightarrow[\theta]{} & \operatorname{Im}\varphi.
\end{array}$$

It is easily verified that $\varphi = \pi_2\theta\iota_1$ and that θ is a bijection. Thus, if $\psi = \pi_1\theta^{-1}\iota_2$, then since $\iota_1\pi_1 = \iota_{\operatorname{Im}\varphi}$ and $\iota_2\pi_2 = \iota_M$, it follows that

$$\varphi\psi\varphi = \pi_2\theta\iota_1\pi_1\theta^{-1}\iota_2\pi_2\theta\iota_1 = \pi_2\theta\iota_1 = \varphi.$$

That is, φ is regular and (1.2) implies (1.1).

It is to be observed, however, that if $\rho\colon F \to \operatorname{Im}\varphi$ and $\sigma\colon M \to E$ are any two R-morphisms such that $\iota_1\rho = \iota_{\operatorname{Im}\varphi}$ and $\sigma\pi_2 = \iota_M$, then $\varphi(\rho\theta^{-1}\sigma)\varphi = \varphi$. That is, there is in general more than one $\psi\colon F \to E$ such that $\varphi\psi\varphi = \varphi$. But, if $\psi = \pi_1\theta^{-1}\iota_2$, then

$$\psi\varphi\psi = \pi_1\theta^{-1}\iota_2\pi_2\theta\iota_1\pi_1\theta^{-1}\iota_2 = \pi_1\theta^{-1}\iota_2 = \psi$$

is also satisfied. Indeed, we have the following corollary.

COROLLARY 1. *If $\varphi\colon E \to F$ is an R-morphism and $E = \operatorname{Ker}\varphi \oplus M$, $F = \operatorname{Im}\varphi \oplus N$, then there exists one and only one R-morphism $\psi\colon F \to E$ such that*

$$\varphi\psi\varphi = \varphi, \qquad \psi\varphi\psi = \varphi, \qquad \operatorname{Im}\psi = M, \qquad \operatorname{Ker}\psi = N.$$

Proof. The existence is clear from the preceding results. Thus, suppose $\psi'\colon F \to E$ is also such that $\varphi\psi'\varphi = \varphi$, $\psi'\varphi\psi' = \psi'$, $\operatorname{Im}\psi' = M$, and $\operatorname{Ker}\psi' = N$. Since $x = x\varphi\psi' + (x - x\varphi\psi')$ is such that $x\varphi\psi' \in \operatorname{Im}\varphi\psi' = \operatorname{Im}\psi' = M$ and $x - x\varphi\psi' \in \operatorname{Ker}\varphi$, it follows that $\varphi\psi' = \pi_2\iota_2$. Similarly, $\psi'\varphi = \pi_1\iota_1$. Consequently

$$\psi' = \psi'\varphi\psi' = \psi'\pi_2\iota_2 = \psi'\varphi\psi = \pi_1\iota_1\psi = \psi\varphi\psi = \psi,$$

and the proof is complete.

The unique R-morphism ψ of this corollary is called the *generalized inverse* of φ relative to the given direct sum decomposition of E and F. It is clear that $\varphi\psi\colon E \to E$ is the projection on $\operatorname{Im}\psi$ along $\operatorname{Ker}\varphi$ and $\psi\varphi\colon F \to F$ is the projection on $\operatorname{Im}\varphi$ along $\operatorname{Ker}\psi$.

We now consider the question: under what conditions on R is every R-morphism regular?

THEOREM 2. *The following statements are equivalent*:

(2.1) *Every R-morphism is regular.*

(2.2) *Every short exact sequence of R-morphisms splits.*

Proof. Suppose (2.1) holds and consider the short exact sequence

$$0 \to D \to E \xrightarrow{\varphi} F \to 0.$$

Let $\psi: F \to E$ be such that $\varphi\psi\varphi = \varphi$. Since φ is surjective, if $y \in F$, then $y = x\varphi$ for some $x \in E$ and

$$y(\psi\varphi) = x\varphi(\psi\varphi) = x(\varphi\psi\varphi) = x\varphi = y.$$

That is, $\psi\varphi = \iota_F$ and the sequence splits.

Conversely, suppose (2.2) holds and consider the R-morphism $\varphi: E \to F$. Then both the vertical and the horizontal sequences of the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & \\
 & \operatorname{Ker}\varphi \arrow[d, "i"] & & & & \\
 & E \arrow[d, shift left, "\varphi_1"] \arrow[drr, "\varphi"] & & & & \\
0 \arrow[r] & \operatorname{Im}\varphi \arrow[u, shift left, "\nu"] \arrow[rr, shift left, "\iota_1"] \arrow[d] & & F \arrow[ll, shift left, "\mu"] \arrow[r] & F/\operatorname{Im}\varphi \arrow[r] & 0 \\
 & 0 & & & &
\end{tikzcd}$$

are short exact sequences, where $\iota: \operatorname{Ker}\varphi \to E$ and $\iota_1: \operatorname{Im}\varphi \to F$ are the natural injections and $\varphi_1: E \to \operatorname{Im}\varphi$ is induced by φ. By (2.2) let $\mu: F \to \operatorname{Im}\varphi$ and $\nu: \operatorname{Im}\varphi \to E$ be such that $\iota_1\mu = \iota_{\operatorname{Im}\varphi}$ and $\nu\varphi_1 = \iota_{\operatorname{Im}\varphi}$. Consequently, since $\varphi = \varphi_1\iota_1$,

$$\varphi(\mu\nu)\varphi = (\varphi_1\iota_1)(\mu\nu)(\varphi_1\iota_1) = \varphi_1(\iota_1\mu)(\nu\varphi_1)\iota_1 = \varphi_1\iota_1 = \varphi,$$

and φ is regular. (See also [11].)

COROLLARY 2. *R is a semi-simple Artinian ring if and only if every R-morphism is regular.*

Proof. See for example [15, p. 12].

One of the important applications of generalized inverses has been to provide "virtual" or "best approximate" solutions to linear systems.

We now specialize our present study to inner product spaces and demonstrate that the existence of particular virtual solutions is in fact equivalent to the existence of special generalized inverses. For convenience we consider real ($R = \mathbb{R}$) inner product spaces (see for example [9]), but it is clear that the results extend to other inner product spaces as well.

For a real inner product space E, denote the symmetric, positive definite bilinear product by (x, y), the norm by $|x| = \sqrt{(x, x)}$, and the subspace orthogonal to a given subspace E_1 of E by $E_1{}^{\perp}$. If $\varphi: E \to F$ is an $\mathbb{R}$-morphism of two real inner product spaces, let φ^* denote the adjoint, provided it exists.

Suppose E_1 is a subspace of a real inner product space E and x is a vector of E. Although there need not exist a vector of E_1 that is "closest" to x, if such a vector x_1 exists, then $x - x_1$ belongs to $E_1{}^{\perp}$. (See for example [1, p. 15].) More completely stated, we have the following lemma.

LEMMA 1. *Let E be a real inner product space, E_1 a subspace, $x \in E$,*

$$\delta = \inf_{x_1 \in E_1} |x - x_1|,$$

and $p \in E_1$. Then $\delta = |x - p|$ if and only if $x - p \in E_1{}^{\perp}$. In particular, there is at most one $p \in E_1$ such that $|x - p| = \delta$.

Proof. Suppose $x - p \in E_1{}^{\perp}$. Then, for any $x_1 \in E_1$,

$$x - x_1 = (p - x_1) + (x - p), \qquad p - x_1 \in E_1, \qquad x - p \in E_1{}^{\perp}.$$

Therefore,

$$|x - x_1|^2 = |p - x_1|^2 + |x - p|^2 \geqslant |x - p|^2.$$

Consequently, $|x - p| \leqslant |x - x_1|$ and $|x - p| = \delta$.

Conversely, suppose $x - p \notin E_1{}^{\perp}$. Let $q \in E_1$ be such that $\gamma = (x - p, q)/(q, q)$ is not zero, and consider $z = p + \gamma q$. Then

$$\begin{aligned} |x - z|^2 &= (x - p - \gamma q, x - p - \gamma q) = |x - p|^2 - 2\gamma(x - p, q) + \gamma^2(q, q) \\ &= |x - p|^2 - \gamma^2(q, q) < |x - p|^2. \end{aligned}$$

Hence, since $z \in E_1$ and $|x - z| < |x - p|$, $\delta \neq |x - p|$.

In particular, it follows that there is at most one vector of E_1 that is closest to x. For if $x_1, x_1' \in E_1$ are such that $x - x_1, x - x_1' \in E_1{}^{\perp}$, then $x_1 - x_1' \in E_1 \cap E_1{}^{\perp} = 0$.

DEFINITION. Let E and F be real inner product spaces, let $\varphi\colon E \to F$ be an $\mathbb{R}$-morphism, and let $d \in F$. Then $b \in E$ is said to be a *virtual solution* of $x\varphi = d$ provided $|d - b\varphi| = \inf_{x \in E} |d - x\varphi|$. Let $d\varphi^{\sim}$ denote the collection of virtual solutions of $x\varphi = d$. If $d\varphi^{\sim}$ is not empty and $b_0 \in d\varphi^{\sim}$ is such that $|b_0| = \inf_{b \in d\varphi^{\sim}} |b|$, then b_0 is said to be an *extremal virtual solution* of $x\varphi = d$.

In general, virtual solutions need not exist. For the convenience of the reader, two simple examples of this fact are now given. These examples may also prove helpful as illustrations of the theorem that follows.

Let $E = F = \mathbb{R}^{(N)}$ be the space of infinite sequences of real numbers, only a finite number of which are not zero. For $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ define $(x, y) = \sum x_i y_i$. In the first case, if

$$\varphi\colon E \to F;\ (x_1, x_2, \ldots) \to (\textstyle\sum x_i, x_1, x_2, \ldots),$$

then, in particular, $(1, 0, 0, \ldots)\varphi^{\sim}$ is empty. In the second case, if

$$\varphi\colon E \to F;\ (x_1, x_2, \ldots) \to (\textstyle\sum x_i, 0, 0, \ldots),$$

then there is no extremal virtual solution of $x\varphi = (1, 0, 0, \ldots)$, even though virtual solutions do exist.

LEMMA 2. *Under the conditions of the previous definition,*

(2.1) *$d\varphi^{\sim}$ is either empty or is a parallel (coset) of the kernel of φ;*
(2.2) *$b \in d\varphi^{\sim}$ if and only if $d - b\varphi \in (\operatorname{Im} \varphi)^{\perp}$;*
(2.3) *If $b \in d\varphi^{\sim}$, then b is an extremal virtual solution of $x\varphi = d$ if and only if $b \in (\operatorname{Ker} \varphi)^{\perp}$;*
(2.4) *there is at most one extremal virtual solution of $x\varphi = d$.*

Proof. (2.1) Suppose $b \in d\varphi^{\sim}$. We show that

$$d\varphi^{\sim} = b + \operatorname{Ker} \varphi.$$

First, let $b + k \in b + \operatorname{Ker} \varphi$. Then $|d - (b + k)\varphi| = |d - b\varphi|$ and $b + k \in d\varphi^{\sim}$. Second, let $b' \in d\varphi^{\sim}$. Since, by Lemma 1, there is at most one vector of $\operatorname{Im} \varphi$ closest to d, $b\varphi = b'\varphi$ and $b' \in b + \operatorname{Ker} \varphi$.

(2.2) Since $\operatorname{Im} \varphi$ is a subspace of F, this result is an immediate application of Lemma 1.

(2.3) Let $b \in d\varphi^{\sim}$ and let b_0 be a vector in $d\varphi^{\sim} = b + \operatorname{Ker} \varphi$ whose distance from 0 is the shortest. This implies that 0 is the vector of $\operatorname{Ker} \varphi$

whose distance from b_0 is the shortest: for if $k \in \operatorname{Ker} \varphi$, then since $b_0 - b \in \operatorname{Ker} \varphi$,

$$|b_0 - 0| \leqslant |(b + ((b_0 - b) - k)) - 0| = |b_0 - k|.$$

Consequently, by Lemma 1, $b_0 = b_0 - 0 \in (\operatorname{Ker} \varphi)^{\perp}$. Conversely, let $b_0 \in d\varphi^{\sim}$ with $b_0 \in (\operatorname{Ker} \varphi)^{\perp}$. Then $b_0 - 0 \in (\operatorname{Ker} \varphi)^{\perp}$ with $0 \in \operatorname{Ker} \varphi$. Thus, by Lemma 1, $|b_0| = |b_0 - 0| \leqslant |b_0 - k|$ for any $k \in \operatorname{Ker} \varphi$, which requires by (2.1) above that $|b_0| \leqslant |b|$ for any $b \in d\varphi^{\sim}$. That is, b_0 is an extremal virtual solution of $x\varphi = d$.

(2.4) If b_0, b_0' are extremal virtual solutions of $x\varphi = d$, since $b_0 - b_0'$ is in both $\operatorname{Ker} \varphi$ and $(\operatorname{Ker} \varphi)^{\perp}$, $b_0' = b_0$.

THEOREM 3. *If E and F are real inner product spaces and $\varphi: E \to F$ is an $\mathbb{R}$-morphism, then the following statements are equivalent.*

(3.1) $E = \operatorname{Ker} \varphi \oplus (\operatorname{Ker} \varphi)^{\perp}$, $F = \operatorname{Im} \varphi \oplus (\operatorname{Im} \varphi)^{\perp}$.

(3.2) *There exists an $\mathbb{R}$-morphism $\varphi^{\dagger}: F \to E$ such that*

$$\varphi\varphi^{\dagger}\varphi = \varphi, \quad \varphi^{\dagger}\varphi\varphi^{\dagger} = \varphi^{\dagger}, \quad (\varphi\varphi^{\dagger})^{*} = \varphi\varphi^{\dagger}, \quad (\varphi^{\dagger}\varphi)^{*} = \varphi^{\dagger}\varphi.$$

(3.3) *There exists an extremal virtual solution of $x\varphi = d$ for every $d \in F$.*

Proof. (3.1) $\to$ (3.2). By Corollary 1, (3.1) implies the existence of $\varphi^{\dagger}: F \to E$ such that

$$\varphi\varphi^{\dagger}\varphi = \varphi, \quad \varphi^{\dagger}\varphi\varphi^{\dagger} = \varphi^{\dagger}, \quad \operatorname{Im} \varphi^{\dagger} = (\operatorname{Ker} \varphi)^{\perp}, \quad \operatorname{Ker} \varphi^{\dagger} = (\operatorname{Im} \varphi)^{\perp}.$$

Therefore, $\varphi\varphi^{\dagger}$ is the projection on $(\operatorname{Ker} \varphi)^{\perp}$ along $\operatorname{Ker} \varphi$ and $\varphi^{\dagger}\varphi$ is the projection on $\operatorname{Im} \varphi$ along $(\operatorname{Im} \varphi)^{\perp}$. But this requires that $\varphi\varphi^{\dagger}$ and $\varphi^{\dagger}\varphi$ are each their own adjoints, and (3.2) is satisfied.

(3.2) $\to$ (3.3). Let $d \in F$ and assume (3.2). Since for any $x \in E$,

$$(d(\iota_F - \varphi^{\dagger}\varphi), x\varphi) = (d, x\varphi(\iota_F - \varphi^{\dagger}\varphi)) = 0,$$

$d - (d\varphi^{\dagger})\varphi \in (\operatorname{Im} \varphi)^{\perp}$. Hence by (2.2) Lemma 2, $d\varphi^{\dagger} \in d\varphi^{\sim}$.

Moreover, if $k \in \operatorname{Ker} \varphi$, then $(d\varphi^{\dagger}, k) = (d\varphi^{\dagger}\varphi\varphi^{\dagger}, k) = (d\varphi^{\dagger}, k\varphi\varphi^{\dagger}) = 0$ implies $d\varphi^{\dagger} \in (\operatorname{Ker} \varphi)^{\perp}$. That is, by (2.3), $d\varphi^{\dagger}$ is the extremal virtual solution of $x\varphi = d$, and (3.3) is satisfied.

(3.3) $\to$ (3.1). Let d be an arbitrary vector of F, and under the assumption (3.3), let b be the unique vector of minimum norm in $d\varphi^{\sim}$. By (2.2)

$d - b\varphi \in (\operatorname{Im}\varphi)^{\perp}$. Hence, since $d = b\varphi + (d - b\varphi)$ and $\operatorname{Im}\varphi \cap (\operatorname{Im}\varphi)^{\perp} = 0$, $F = \operatorname{Im}\varphi \oplus (\operatorname{Im}\varphi)^{\perp}$.

Next, let a be an element of E. Again, by (3.3), let b_a be the unique vector of minimum norm in $(a\varphi)\varphi^{\sim}$. Since $a \in (a\varphi)\varphi^{\sim}$, it follows from (2.1) and (2.3) that $a - b_a \in \operatorname{Ker}\varphi$ and $b_a \in (\operatorname{Ker}\varphi)^{\perp}$. Thus, $a = (a - b_a) + b_a$ implies $E = \operatorname{Ker}\varphi \oplus (\operatorname{Ker}\varphi)^{\perp}$, which completes the proof.

COROLLARY 3. *Let the notation and conditions be as in Theorem* 3. *If* (3.1) (*hence also* (3.2) *and* (3.3)) *is satisfied, then*

(i) $\varphi^{\dagger}$ *of* (3.2) *is unique.*
(ii) $d\varphi^{\sim} = d\varphi^{\dagger} + \operatorname{Ker}\varphi$ *for each* $d \in F$.
(iii) $d\varphi^{\dagger}$ *is the extremal virtual solution of* $x\varphi = d$.
(iv) *If* $\{y_a\}$ *is a basis of* F *and* x_a *is the extremal virtual solution of* $x\varphi = y_a$, *then* $\varphi^{\dagger}$ *is the linear extension of* $y_a \to x_a$.

The two examples given above provide examples of where $\varphi^{\dagger}$ of (3.2) fails to exist. This is clear from condition (3.3). Alternatively, the conclusion follows also from an examination of (3.1). In the first case, since $(1, 0, 0, \ldots) \notin \operatorname{Im}\varphi$ and $(\operatorname{Im}\varphi)^{\perp} = 0$, $F \neq \operatorname{Im}\varphi + (\operatorname{Im}\varphi)^{\perp}$. Similarly, in the second case, $(\operatorname{Ker}\varphi)^{\perp} = 0$ and $E \neq \operatorname{Ker}\varphi + (\operatorname{Ker}\varphi)^{\perp}$.

It is evident that Theorem 3 may also be established for complex inner product spaces. In particular, if E and F are complete Hilbert spaces and $\varphi: E \to F$ is a bounded linear transformation, then it is well known that $\operatorname{Ker}\varphi$ is closed and hence $E = \operatorname{Ker}\varphi \oplus (\operatorname{Ker}\varphi)^{\perp}$. However, $\operatorname{Im}\varphi$ need not be closed. We remark in this case that condition (3.1) is equivalent to the condition that $\operatorname{Im}\varphi$ be closed. For if $\operatorname{Im}\varphi$ is closed, then $F = \operatorname{Im}\varphi \oplus (\operatorname{Im}\varphi)^{\perp}$ and (3.1) is satisfied. Conversely, let (3.1) be satisfied and let d be in the closure of $\operatorname{Im}\varphi$. Since $\delta = \inf_{x\in E} |x\varphi - d| = 0$, by (3.3) there exists a b such that $|b\varphi - d| = 0$; that is, $d = b\varphi \in \operatorname{Im}\varphi$ and $\operatorname{Im}\varphi$ is closed. (See for example [5].)

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ADDENDUM (JULY, 1973)

Due to unusual delays in the publication of this paper, the Editor-in-Chief has recommended that some additions be appended.

Reference [2′] listed below is a sequel to the first part of the present paper. There the notion of a generalized inverse of a morphism in an arbitrary category is studied, and Theorem 1 above is included as one application of the results [2′, Theorem 4]. Also, an alternate proof of Theorem 2 above now follows from Corollary 3 of [2′] and some facts about the category of R-modules: viz., a R-morphism is left (right) cancellative if and only if it is surjective (injective) as a function, and every short exact sequence of R-modules splits if and only if every surjective (injective) R-morphism is left (right) invertible. (See [15] above, pp. 8–12.)

Reference [3′] considers generalized inverses in Hilbert space and contains some material that is related to the second part of the present paper. In particular, a necessary and sufficient condition for the existence of an extremal virtual solution of $x\varphi = d$ is given in terms of d and a generalized inverse of the operator φ. Also, examples of equations that fail to possess extremal virtual solutions are included from the space of square-summable sequences and the space of square-integrable functions.

Additional relative references are listed in both [2′] and [3′], and extensive bibliographies on the subject of generalized inverses are given in [1′] and [4′]. Section 2 of [4′] also provides a succinct discussion, which relates to the final paragraph above, on the existence of generalized inverses of bounded linear transformations on Hilbert spaces.

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